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THE HOMOLOGY OF PRINCIPALLY DIRECTED ORDERED GROUPOIDS

B. O. BAINSON AND N.D. GILBERT

Abstract

We present some homological properties of a relation β on ordered groupoids that generalises the minimum group congruence for inverse semigroups. When β is a transitive relation on an ordered groupoid G , the quotient G/β is again an ordered groupoid, and we construct a pair of adjoint functors between the module categories of G and of G/β . As a consequence, we show that the homology of G is completely determined by that of G/β , generalising a result of Loganathan for inverse semigroups.

1. Introduction

This paper studies some homological properties of a quotient construction for ordered groupoids determined by a certain relation β that generalises the minimal group congruence σ on an inverse semigroup. Modules for inverse semigroups, and the cohomology of an inverse semigroup, were first defined by Lausch in [9], and the cohomology used to classify extensions. An approach based on the cohomology of categories was then given by Loganathan [13], who showed that Lausch's cohomology of an inverse semigroup S was equal to the cohomology of a left-cancellative category $\mathfrak{L}(S)$ naturally associated to S . Loganathan proves a number of results relating the cohomology of S with that of its semilattice of idempotents $E(S)$ and of its maximum group image S/σ . He also considers the homology of S , but the treatment is brief since [13, Proposition 3.5] shows that the homology of S is completely determined by the homology of the group S/σ .

Ordered groupoids and inverse semigroups are closely related, since any inverse semigroup can be considered as a particular kind of ordered groupoid – an *inductive* groupoid – and this correspondence in fact gives rise to an isomorphism between the category of inverse semigroups and the category of inductive groupoids. This is the Ehresmann-Schein-Nambooripad Theorem (see [11, Theorem 4.1.8]). This close relationship has been exploited in the use of ordered groupoid techniques to prove results about inverse semigroups (see [7, 11, 12, 16]) and has been the motivation

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behind various generalisations of results about inverse semigroups to the wider class of ordered groupoids (see [1, 5, 10]).

In this paper we revisit Loganathan's results on the homology of inverse semigroups, and we are led to consider the relation β on an ordered groupoid G defined as follows: two elements of G are β -related if and only if they have a lower bound in G . This relation is trivially reflexive and symmetric but need not be transitive: when it is, we say that G is a principally directed ordered groupoid, a choice of terminology justified in Lemma 3.1 below. The β -relation and the class of principally directed ordered groupoids featured in [5] (but there called β -transitive ordered groupoids), in the study of the structure of inverse semigroups S with zero. In this setting, $S^* = S \setminus \{0\}$ can be considered as an ordered groupoid, and S^* is then principally directed if and only if S is *categorical at zero*: that is, whenever $a, b, c \in S$ and $abc = 0$ then either $ab = 0$ or $bc = 0$. The structure theorem of Gomes and Howie [6] for strongly categorical inverse semigroups with zero can then be deduced from a more general result on principally directed ordered groupoids [5, section 4.1]. In this paper, the significance of the transitivity of β is that it permits the construction of a pair of adjoint functors between the module categories of G and of G/β . The left adjoint is simply the colimit over $E(G)$. The right adjoint expands a G/β -module to a G -module. These constructions are discussed in section 4, and generalise the key ingredients of Loganathan's treatment of the homology of inverse semigroups in [13]. The fact that the homology of a principally directed ordered groupoid G is determined by the homology of the quotient G/β then follows readily in section 5.

2. Ordered Groupoids

A groupoid G is a small category in which every morphism is invertible. The set of identities of G is denoted $E(G)$, following the customary notation for the set of idempotents in an inverse semigroup. We write $g \in G(e, f)$ when g is a morphism starting at e and ending at f . We regard a groupoid as an algebraic structure comprising its morphisms, and compositions of morphisms as a partially defined binary operation (see [8], [11]). The identities are then written as $e = g\mathbf{d} = gg^{-1}$ and $f = g\mathbf{r} = g^{-1}g$ respectively. A groupoid map $\theta : G \rightarrow H$ is just a functor.

Definition 2.1. An ordered groupoid is a pair (G, \leq) where G is a groupoid and \leq is a partial order defined on G , satisfying the following axioms:

- OG1 $x \leq y \Rightarrow x^{-1} \leq y^{-1}$, for all $x, y \in G$.
- OG2 Let $x, y, u, v \in G$ such that $x \leq y$ and $u \leq v$. Then $xu \leq yv$ whenever the compositions xu and yv exist.
- OG3 Suppose $x \in G$ and $e \in E(G)$ such that $e \leq x\mathbf{d}$, then there is a unique element $(e|x)$ called the *restriction* of x to e such that $(e|x)\mathbf{d} = e$ and $(e|x) \leq x$.
- OG4 If $x \in G$ and $e \in E(G)$ such that $e \leq x\mathbf{r}$, then there exist a unique element $(x|e)$ called the *corestriction* of x to e such that $(x|e)\mathbf{r} = e$ and $(x|e) \leq x$.

It is easy to see that OG3 and OG4 are equivalent: if OG3 holds then we may define a corestriction $(x|e)$ by $(x|e) = (e|x^{-1})^{-1}$.

An ordered functor $\phi : G \rightarrow H$ of ordered groupoids is an order preserving groupoid-map, that is $g\phi \leq h\phi$ if $g \leq h$. Ordered groupoids together with ordered functors constitute the category of ordered groupoids, **OGpd**.

Suppose $g, h \in G$ and that the greatest lower bound ℓ of $g\mathbf{r}$ and $h\mathbf{d}$ exist, then we define the *pseudoproduct* of g and h by $g * h = (g|\ell)(\ell|h)$. An ordered groupoid is called *inductive* if the pair $(E(G), \leq)$ is a meet semilattice. In an inductive groupoid G , the pseudoproduct is everywhere defined and $(G, *)$ is then an inverse semigroup: see [11, Theorem 4.1.8]

Example 2.2. For any set X , consider the set $\mathfrak{I}(X)$ of all bijections between subsets of X . Then $\mathfrak{I}(X)$ is an ordered groupoid, with set of identities $E(\mathfrak{I}(X))$ equal to the set of identity maps on subsets on X . Two bijections $\alpha : U \rightarrow V$ and $\beta : W \rightarrow Y$ are composable if and only if $V = W$, resulting in the usual composition $\alpha\beta : U \rightarrow Y$. The partial order is given by restriction, so that given $\alpha : U \rightarrow V$ and $\beta : W \rightarrow Y$, we have $\alpha \leq \beta$ if and only if $U \subseteq W$ and, for all $x \in U$, $x\alpha = x\beta$. If we allow the empty map as an element of $\mathfrak{I}(X)$, then the pseudoproduct $*$ is everywhere defined and the inverse semigroup $(\mathfrak{I}(X), *)$ is the *symmetric inverse monoid* on X , see [11, Chapter 1].

To any ordered groupoid G we associate a category $\mathfrak{L}(G)$ as follows. The objects of $\mathfrak{L}(G)$ are the identities of G and morphisms are given by pairs $(e, g) \in E(G) \times G$ where $g\mathbf{d} \leq e$, with $(e, g)\mathbf{d} = e$ and $(e, g)\mathbf{r} = g\mathbf{r}$. The composition of morphisms is defined by the partial product $(e, g)(f, h) = (e, g * h) = (e, (g|h\mathbf{d})h)$ whenever $g\mathbf{r} = f$. It is easy to see that $\mathfrak{L}(G)$ is left cancellative. This construction originates in the work of Loganathan [13], and forms the basis of the treatment in [13] of the cohomology of inverse semigroups.

3. Principally directed ordered groupoids

Let G be an ordered groupoid. The relation β on G is defined by

$$g\beta h \iff \text{there exists } k \in G \text{ with } k \leq g \text{ and } k \leq h.$$

β is evidently reflexive and symmetric but need not be transitive: we shall be concerned with the class of ordered groupoids for which β is indeed transitive, and thus an equivalence relation. We shall denote the β -class of $g \in G$ by $g\beta$. A *principal order ideal* is a subset of G of the form $\{g \in G : g \leq t\}$ for some $t \in G$, and will be denoted by $(t)^\downarrow$.

Lemma 3.1. [5, section 2.2] *The β -relation on an ordered groupoid G is transitive if and only if every principal order ideal in G is a directed set.*

Proof. Suppose that β is transitive, and that $g, h \in (t)^\downarrow$. Then $g\beta t\beta h$ and so $g\beta h$, and there exists $k \in G$ with $k \leq g$ and $k \leq h$: hence $k \in (t)^\downarrow$ and $(t)^\downarrow$ is a directed set. Conversely, suppose that $g\beta t\beta h$: then there exist $k, l \in G$ with $k \leq g$, $k \leq t$, $l \leq t$ and $l \leq h$. In particular, $k, l \in (t)^\downarrow$, and if $(t)^\downarrow$ is a directed set then there exists $c \leq t$ with $c \leq k$ and $c \leq l$. Then $c \leq g$ and $c \leq h$, and so $g\beta h$. \square

Definition 3.2. An ordered groupoid in which every principal order ideal is a directed set will be called *principally directed*. This terminology is consistent with that of [10].

It is clear that if G is principally directed then so is its poset of identities $E(G)$. However, the converse is false. Let A and B be groups with a common subgroup C and let $i : C \hookrightarrow A$ and $j : C \hookrightarrow B$ be the inclusions. Consider the semilattice $\{0, e, f, 1\}$ with e, f incomparable, and define a semilattice of groups G by $G_1 = C, G_e = A, G_f = B$ and $E(G) = A \times B$ and with the obvious structure maps. Then $ci\beta c\beta cj$ for all $c \in C$, but ci and cj are not β -related.

Proposition 3.3. [5, Proposition 2.2] *If G is a principally directed ordered groupoid then the quotient set G/β is a groupoid.*

The groupoid structure on G/β is inherited from G in the following way. If $g, h \in G$ and $g^{-1}g\beta hh^{-1}$ then there exists $f \in E(G)$ with $f \leq g^{-1}g$ and $f \leq hh^{-1}$, and the composition of the β -classes of g and h is then defined by

$$(g\beta)(h\beta) = [(g|f)(f|h)]\beta.$$

This is easily seen to be independent of any choices made for f and for representatives of $g\beta$ and $h\beta$: see [5, section 2.2] for further details. However, there is no natural ordering inherited by G/β , and so we regard G/β as trivially ordered. Lawson [10, Theorem 20] states Proposition 3.3 for the special case of *principally inductive* ordered groupoids.

Example 3.4. An *action* of an ordered groupoid G on a set X is given by an ordered functor $G \rightarrow \mathfrak{J}(X)$ (see Example 2.2). There is an associated *action groupoid* $X \rtimes G$, which is an ordered groupoid defined as follows:

$$X \rtimes G = \{(x, g) \in X \times G : x \in (g\alpha)\mathbf{d}\},$$

with set of identities $E(X \rtimes G) = \{(x, e) \in X \times G : e \in E(G)\}$. We have $(x, g)\mathbf{d} = (x, gg^{-1})$ and $(x, g)\mathbf{r} = (x(g\alpha), g^{-1}g)$, and the composition of (x, g) and (y, h) is defined, when $y = x(g\alpha)$ and $g^{-1}g = hh^{-1}$, by $(x, g)(y, h) = (x, gh)$. In the case that $X \rtimes G$ is principally directed, then $(X \rtimes G)/\beta$ is the *groupoid of germs* of the action. This holds when G is inductive, giving the construction of [3, Example 3.5]: see also [14, Chapter 1, p15] and [10, section 8] for actions of G by partial homeomorphisms of a topological space.

4. Expansion and colimits of modules

Let G be an ordered groupoid, and $\mathfrak{L}(G)$ its associated left-cancellative category. A G -module is defined to be an $\mathfrak{L}(G)$ -module, that is, a functor \mathcal{A} from $\mathfrak{L}(G)$ to the category of abelian groups. A G -module \mathcal{A} is thus comprised of a family of abelian groups $\{A_e : e \in E(G)\}$ together with a group homomorphism $\alpha_{(e,g)} : A_e \rightarrow A_{g^{-1}g}$ for each arrow (e, g) of $\mathfrak{L}(G)$. We shall often denote $a\alpha_{(e,g)}$ by $a \triangleleft (e, g)$. Morphisms of G -modules (called *G -maps*) are natural transformations of functors, and so we obtain a category Mod_G of G -modules and G -maps.

Suppose that G is principally directed. No ordering is prescribed for the quotient groupoid G/β and so $\mathfrak{L}(G/\beta) = G/\beta$. If \mathcal{B} is a (G/β) -module then we can *expand* \mathcal{B} to obtain an $\mathfrak{L}(G)$ -module $\mathcal{B}_\beta^\uparrow$ with homomorphisms $\mu_{(e,g)}$ as follows:

- for $e \in E(G)$ we have $(\mathcal{B}_\beta^\uparrow)_e = B_{e\beta}$,

- if $e \geq f$ then $e\beta = f\beta$ and $\mu_{(e,f)} = \text{id}$,
- for $x, y \in E(G)$ and for each $g \in G(x, y)$, the map $\mu_{(x,g)} : B_{x\beta} \rightarrow B_{y\beta}$ is just the map $\mu_{g\beta} : B_{x\beta} \rightarrow B_{y\beta}$ determined by \mathcal{B} .

This defines the *expansion functor* $\text{Mod}_{G/\beta} \rightarrow \text{Mod}_{\mathfrak{L}(G)}$ since, if $\xi : \mathcal{B} \rightarrow \mathcal{B}'$ is a G/β -map then we have a commutative diagram

$$\begin{array}{ccc}
 B_{e\beta} & \xrightarrow{\xi_{e\beta}} & B'_{e\beta} \\
 \parallel & & \parallel \\
 B_{(gg^{-1})\beta} & \xrightarrow{\xi_{e\beta}} & B'_{(gg^{-1})\beta} \\
 \downarrow \triangleleft g\beta & & \downarrow \triangleleft g\beta \\
 B_{(g^{-1}g)\beta} & \xrightarrow{\xi_{(g^{-1}g)\beta}} & B'_{(g^{-1}g)\beta}
 \end{array}$$

$\triangleleft(e,g)$

and so we obtain an $\mathfrak{L}(G)$ -map $\xi_\beta^\uparrow : \mathcal{B}_\beta^\uparrow \rightarrow (\mathcal{B}')_\beta^\uparrow$ with $(\xi_\beta^\uparrow)_e = \xi_{e\beta}$.

Lemma 4.1. *The expansion functor $\text{Mod}_{G/\beta} \rightarrow \text{Mod}_{\mathfrak{L}(G)}$ preserves epimorphisms.*

Proof. Epimorphisms in Mod are given by families of surjections, and so if ξ is an epimorphism in $\text{Mod}_{G/\beta}$ then so is ξ_β^\uparrow in $\text{Mod}_{\mathfrak{L}(G)}$. \square

The expansion functor is implicit in [13] for the case in which β is replaced by the minimal group congruence σ on an inverse semigroup. We now generalise [13, Lemma 3.4] and show that the expansion functor $\text{Mod}_{G/\beta} \rightarrow \text{Mod}_{\mathfrak{L}(G)}$ for a principally directed ordered groupoid G admits a left adjoint.

Suppose that \mathcal{A} is an $\mathfrak{L}(G)$ -module. We consider the restriction of \mathcal{A} to an $E(G)$ -module, involving the same abelian groups A_e , ($e \in E(G)$) but using only the maps $\alpha_{(e,f)} : A_e \rightarrow A_f$ from \mathcal{A} . The colimit $\text{colim}^{E(G)} \mathcal{A}$ is then a direct sum

$$\text{colim}^{E(G)} \mathcal{A} = \bigoplus_{x \in E(G)/\beta} L_x$$

indexed by the β -classes in $E(G)$, and so determines an $E(G/\beta)$ -module \mathcal{L} with $C_{e\beta} = L_{e\beta}$ and with trivial action, since $E(G/\beta)$ is a trivially ordered poset. We shall allow ourselves a small abuse of notation, and denote \mathcal{L} by $\text{colim}^{E(G)} \mathcal{A}$.

Proposition 4.2. *If G is principally directed and \mathcal{A} is a G -module then $\text{colim}^{E(G)} \mathcal{A}$ is a G/β -module.*

Proof. Let $\text{colim}^{E(G)} \mathcal{A} = \bigoplus L_x$ as above, let $\alpha_e : A_e \rightarrow L_{e\beta}$ be the canonical map. Suppose that $\bar{a} \in L_{e\beta}$ with $\bar{a} = a\alpha_e$ for some $a \in A_e$, and $g \in G$ with $gg^{-1}\beta e$. Then gg^{-1} and e have a lower bound ℓ , and we define an action of $g\beta$ on \bar{a} by

$$\bar{a} \triangleleft g\beta = (a\alpha_{(e,\ell)}) \triangleleft (\ell|g)\alpha_z \quad (1)$$

where $z = (\ell|g)\mathbf{r}$. We have to check that this definition is independent of the choices made for ℓ, a and g .

If we choose a different lower bound ℓ' of gg^{-1} and e , then ℓ and ℓ' are β -related (using the transitivity of β) and so have a lower bound ℓ'' . It is sufficient to show, for

independence from the choice of ℓ , that the outcome of (1) is unchanged by descent in the partial order, in the following sense.

Suppose that $a \in A_e$, $gg^{-1} = e$ and that $f \leq e$. Let $y = g^{-1}g$ and $z = (f|g)\mathbf{r}$. Then (1) gives $a\alpha_e \triangleleft g\beta = (a \triangleleft g)\alpha_y$. If we base the calculation at f we obtain $(a\alpha_{(e,f)} \triangleleft (f|g))\alpha_z$. But in $\mathfrak{L}(G)$,

$$(e, f)(f, (f|g)) = (e, (f|g)) = (e, (e|g))(y, z)$$

and so $a\alpha_{(e,f)} \triangleleft (f|g) = (a \triangleleft g)\alpha_{(y,z)}$. Hence

$$(a\alpha_{(e,f)} \triangleleft (f|g))\alpha_z = (a \triangleleft g)\alpha_{(y,z)}\alpha_z = (a \triangleleft g)\alpha_y.$$

Therefore the outcome of (1) is independent of the choice of ℓ .

We now consider the choice of a preimage for \bar{a} . Suppose that $a\alpha_e = b\alpha_x$. Then $e\beta x$ and so e and x have a lower bound u with $\bar{a} = a\alpha_{(e,u)}\alpha_u = b\alpha_{(x,u)}\alpha_u$. So again it suffices to check what happens if we apply (1) at u . We have

$$\bar{a} \triangleleft g\beta = (a \triangleleft g)\alpha_y = (a\alpha_{(e,u)} \triangleleft (u|g))\alpha_z$$

where now $z = (u|g)\mathbf{r}$. But as before, $a\alpha_{(e,u)} \triangleleft (u|g) = (a \triangleleft g)\alpha_{(y,z)}$ and $\alpha_{(y,z)}\alpha_z = \alpha_y$. Hence the definition in (1) is independent of the choice of a .

Finally, suppose that $g\beta h$. Then $gg^{-1}\beta hh^{-1}$ and so gg^{-1} and hh^{-1} have a lower bound $v \in E(G)$. Then $g\beta = (v|g)\beta = (v|h)\beta = h\beta$, and acting with $(v|g)$ in (1) we obtain

$$\begin{aligned} \bar{a} \triangleleft (v|g)\beta &= (a\alpha_v^e \triangleleft (v|g))\alpha_z \\ &= (a \triangleleft g)\alpha_{(y,z)}\alpha_z \\ &= (a \triangleleft g)\alpha_y. \end{aligned}$$

Hence the definition in (1) is independent of the choice of g , and we have a well-defined action of G/β on $\text{colim}^{E(G)} \mathcal{A}$. \square

Let \mathcal{B} be a G/β -module, let \mathcal{A} be a G -module, and suppose that we are given a map $\phi : \mathcal{A} \rightarrow \mathcal{B}_\beta^\uparrow$, with components $\phi_e : A_e \rightarrow B_{e\beta}$, ($e \in E(G)$). Whenever $e \geq f$ we have a commutative triangle

$$\begin{array}{ccc} A_e & \xrightarrow{\alpha_{(e,f)}} & A_f \\ & \searrow \phi_e \quad \swarrow \phi_f & \\ & B_{e\beta} & \end{array}$$

(in which $B_{e\beta} = B_{f\beta}$) and so the ϕ_e induce a family of maps ψ with $\psi_{e\beta} : L_{e\beta} \rightarrow \mathcal{B}_{e\beta}$ and, if $\alpha_e : A_e \rightarrow \text{colim}^{E(G)} \mathcal{A}$ is the canonical map, then $\phi_e = \alpha_e \psi_{e\beta}$. Therefore ψ determines ϕ , and we have the following Corollary of Proposition 4.2.

Corollary 4.3. *If G is principally directed then $\psi : \text{colim}^{E(G)} \mathcal{A} \rightarrow \mathcal{B}$ is a G/β -map, and $\phi \mapsto \psi$ is an injection*

$$\rho : \text{Mod}_{\mathfrak{L}(G)}(\mathcal{A}, \mathcal{B}_\beta^\uparrow) \rightarrow \text{Mod}_{G/\beta}(\text{colim}^{E(G)} \mathcal{A}, \mathcal{B}). \quad (2)$$

Theorem 4.4. *Let G be a principally directed ordered groupoid. Then the functor $\text{colim}^{E(G)} : \text{Mod}_G \rightarrow \text{Mod}_{G/\beta}$ is left adjoint to the expansion functor.*

Proof. We wish to construct a function

$$\tau : \text{Mod}_{G/\beta}(\text{colim}^{E(G)} \mathcal{A}, \mathcal{B}) \rightarrow \text{Mod}_G(\mathcal{A}, \mathcal{B}_\beta^\uparrow). \quad (3)$$

that will be inverse to ρ in (2). For $e \in E(G)$ and $\psi : \text{colim}^{E(G)} \mathcal{A} \rightarrow \mathcal{B}$, consider the composition

$$A_e \xrightarrow{\alpha_e} L_{e\beta} \xrightarrow{\psi_{e\beta}} B_{e\beta} = (\mathcal{B}_\beta^\uparrow)_e.$$

This composition is a G -map since, for $a \in A_e$,

$$(a\alpha_e\psi)\mu_{g\beta} = (a\alpha_e \triangleleft g\beta)\psi_{(g^{-1}g)\beta}$$

and, evaluating the $g\beta$ action using (1) with $\ell = gg^{-1}$,

$$= (a \triangleleft (e, g))\alpha_{(g^{-1}g)\beta}\psi_{(g^{-1}g)\beta}$$

and so the diagram

$$\begin{array}{ccccc} A_e & \xrightarrow{\alpha_e} & L_{e\beta} & \xrightarrow{\psi_{e\beta}} & B_{e\beta} \\ \triangleleft(e, g) \downarrow & & \triangleleft g\beta \downarrow & & \triangleleft g\beta \downarrow \\ A_{g^{-1}g} & \xrightarrow{\alpha_{g^{-1}g}} & L_{(g^{-1}g)\beta} & \xrightarrow{\psi_{(g^{-1}g)\beta}} & B_{(g^{-1}g)\beta} \end{array}$$

commutes. Now the injection ρ in (2) carries $(\alpha_e\psi_{e\beta})$ to ψ and so $\tau\rho$ is the identity. A G -map $\phi : \mathcal{A} \rightarrow \mathcal{B}_\beta^\uparrow$ is carried by ρ to the induced map $\psi : \mathcal{L} \rightarrow \mathcal{B}$, where $\phi_e = \alpha_e\psi_{e\beta}$. But τ carries ψ precisely to this composition, and so $\rho\sigma$ is also the identity, and so in the principally directed case, (2) and (3) exhibit a natural bijection and its inverse. \square

4.1. Composition of colimits

If G is principally directed, then we have seen in Proposition 4.2 that, for every G -module \mathcal{A} , the colimit $\mathcal{L} = \text{colim}^{E(G)} \mathcal{A}$ can be considered as a G/β -module. Since G/β need not be connected, $\text{colim}^{G/\beta} \mathcal{L}$ decomposes in general into a direct sum $\text{colim}^{G/\beta} \mathcal{L} = \bigoplus_{p \in \pi_0(G/\beta)} C_p$ indexed by the connected components of G/β . We can therefore form $\text{colim}^{G/\beta} \mathcal{L}$, with canonical maps $\psi_{e\beta} : L_{e\beta} \rightarrow C_{e\beta}$, where $e\beta$ is the connected component of $e \in G$ in the quotient groupoid G/β .

Proposition 4.5. *The colimit $\text{colim}^{G/\beta}(\text{colim}^{E(G)} \mathcal{A})$ is naturally isomorphic to $\text{colim}^{\mathfrak{L}(G)} \mathcal{A}$.*

Proof. We show that $\text{colim}^{G/\beta} \mathcal{L}$ has the universal property required of $\text{colim}^{\mathfrak{L}(G)} \mathcal{A}$. As above, we have $\alpha_e : A_e \rightarrow L_{e\beta}$ and a commutative diagram

$$\begin{array}{ccccc} A_e & \xrightarrow{\alpha_e} & L_{e\beta} & & \\ \triangleleft(e, g) \downarrow & & \triangleleft g\lambda \downarrow & \searrow \psi_{e\beta} & \\ A_{g^{-1}g} & \xrightarrow{\alpha_{g^{-1}g}} & L_{(g^{-1}g)\beta} & \xrightarrow{\psi_{(g^{-1}g)\beta}} & \text{colim}^{G/\beta} \mathcal{L} \end{array}$$

from which we extract the commutative triangles

$$\begin{array}{ccc}
 A_e & \xrightarrow{\alpha_e \psi_{e\beta}} & \operatorname{colim}^{G/\beta} \mathcal{L} \\
 \downarrow \triangleleft(e,g) & & \uparrow \alpha_{g^{-1}g} \psi_{(g^{-1}g)\beta} \\
 A_{g^{-1}g} & &
 \end{array}$$

Suppose we are given a family of maps $\mu_e : A_e \rightarrow M$ to some abelian group M making commutative triangles

$$\begin{array}{ccc}
 A_e & \xrightarrow{\mu_e} & M \\
 \downarrow \triangleleft(e,g) & & \uparrow \mu_{g^{-1}g} \\
 A_{g^{-1}g} & &
 \end{array}$$

In particular, for $f \leq e$ we have

$$\begin{array}{ccc}
 A_e & \xrightarrow{\mu_e} & M \\
 \downarrow \alpha_{(e,f)} & & \uparrow \mu_f \\
 A_f & &
 \end{array}$$

and hence a unique family of maps $\delta_{e\beta} : L_{e\beta} \rightarrow M$ making the diagrams

$$\begin{array}{ccccc}
 A_e & & & & \\
 \downarrow \alpha_{(e,f)} & \searrow \alpha_e & & \searrow \mu_e & \\
 & & L_{e\beta} & \xrightarrow{\delta_{e\beta}} & M \\
 & \nearrow \alpha_f & & \nearrow \mu_f & \\
 A_f & & & &
 \end{array}$$

commute.

Now consider the action of $g\beta$ on $\bar{a} = a\alpha_e \in L_{e\beta}$. From (1)

$$\begin{aligned}
 (\bar{a} \triangleleft g\beta)\delta_{(g^{-1}g)\beta} &= (a\alpha_e \triangleleft (\ell|g))\alpha_z\delta_{z\beta} \\
 &= (a\alpha_e \triangleleft (\ell|g))\mu_z \\
 &= a\mu_e \quad (\text{since } \mu_e = \alpha_{(e,(\ell,g))}\mu_{g^{-1}g}) \\
 &= a\alpha_e\delta_{e\beta} \\
 &= \bar{a}\delta_{e\beta}.
 \end{aligned}$$

Hence the triangles

$$\begin{array}{ccc}
 L_{e\beta} & & \\
 \downarrow \triangleleft g\beta & \searrow \delta_{e\beta} & \\
 & & M \\
 & \nearrow \delta_{z\beta} & \\
 L_{z\beta} & &
 \end{array}$$

commute and induce a unique map $\delta : \text{colim}^{G/\beta} \mathcal{L} \rightarrow M$ making the diagram

$$\begin{array}{ccccc}
 A_e & \xrightarrow{\alpha_e} & L_{e\beta} & \xrightarrow{\delta_{[e]}} & \\
 \downarrow \triangleleft(e,g) & & \downarrow \triangleleft g\lambda & \searrow \psi_{[e]} & \\
 A_{g^{-1}g} & \xrightarrow{\alpha_{g^{-1}g}} & L_{(g^{-1}g)\beta} & \xrightarrow{\delta_{(g^{-1}g)\beta}} & \\
 & & & \nearrow \psi_{(g^{-1}g)\beta} & \\
 & & & \text{colim}^{G/\beta} \mathcal{L} & \xrightarrow{\delta} M
 \end{array}$$

commute, since $L_{z\beta} = L_{(g^{-1}g)\beta}$. \square

5. The homology of principally directed ordered groupoids

The functors $H_n(G, -)$, $n \geq 0$, for a fixed ordered groupoid G (or equivalently, for the left-cancellative category $\mathfrak{L}(G)$), may be characterized as functors $\text{Mod}_G \rightarrow \mathbf{Ab}$ by the following properties:

- (a) $H_n(G, -)$, $n \geq 0$ is a homological extension of the colimit $\text{colim}^{\mathfrak{L}(G)}$, so that
 - $H_0(G, \mathcal{A}) = \text{colim}^{\mathfrak{L}(G)}(\mathcal{A})$,
 - for any short exact sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ of G -modules and for each $n \geq 0$, there exists a natural homomorphism $d_n : H_{n+1}(G, \mathcal{C}) \rightarrow H_n(G, \mathcal{A})$ inducing an exact sequence

$$\dots \rightarrow H_{n+1}(G, \mathcal{C}) \rightarrow H_n(G, \mathcal{A}) \rightarrow H_n(G, \mathcal{B}) \rightarrow H_n(G, \mathcal{C}) \rightarrow H_{n-1}(G, \mathcal{A}) \rightarrow \dots$$
- (b) $H_n(G, \mathcal{P}) = 0$ for all $n > 0$ and all projective modules \mathcal{P} .

Theorem 5.1. *For any principally directed ordered groupoid G and G -module \mathcal{A} , and any $n \geq 0$, the homology groups $H_n(G, \mathcal{A})$ and $H_n(G/\beta, \text{colim}^{E(G)} \mathcal{A})$ are isomorphic.*

Proof. We consider the functor $\text{Mod}_{\mathfrak{L}(G)} \rightarrow \mathbf{Ab}$ given by

$$\mathcal{A} \mapsto H_n(G/\beta, \text{colim}^{E(G)} \mathcal{A}).$$

For $n = 0$ we have

$$H_0(G/\beta, \text{colim}^{E(G)} \mathcal{A}) = \text{colim}^{G/\beta}(\text{colim}^{E(G)} \mathcal{A}) \cong \text{colim}^{\mathfrak{L}(G)} \mathcal{A} = H_0(G, \mathcal{A})$$

by Proposition 4.5. The transitivity of β on $E(G)$ is sufficient to ensure that $\mathcal{A} \mapsto \text{colim}^{E(G)} \mathcal{A}$ is exact, (see, for example, [15, tag 04AX]). It follows that the sequence of functors $H_n(G/\beta, \text{colim}^{E(G)} -)$ induces, from a short exact sequence $\mathcal{A} \rightarrow \mathcal{B} \rightarrow \mathcal{C}$ of G -modules an exact sequence

$$\begin{aligned} \dots \rightarrow H_{n+1}(G/\beta, \text{colim}^{E(G)} \mathcal{C}) &\rightarrow H_n(G/\beta, \text{colim}^{E(G)} \mathcal{A}) \rightarrow H_n(G/\beta, \text{colim}^{E(G)} \mathcal{B}) \\ &\rightarrow H_n(G/\beta, \text{colim}^{E(G)} \mathcal{C}) \rightarrow H_{n-1}(G/\beta, \text{colim}^{E(G)} \mathcal{A}) \rightarrow \dots \end{aligned}$$

Now suppose that \mathcal{P} is a projective $\mathfrak{L}(G)$ -module. By Lemma 4.1 the expansion functor $\text{Mod}_{G/\beta} \rightarrow \text{Mod}_{\mathfrak{L}(G)}$ preserves epimorphisms, and so its left adjoint $\text{colim}^{E(G)}$ preserves projectives. Therefore $\text{colim}^{E(G)} \mathcal{P}$ is projective, and for $n > 0$ we have $H_n(G/\beta, \text{colim}^{E(G)} \mathcal{P}) = 0$. \square

We note that the homology of groupoids is discussed in [8, chapter 16] (mostly for integer coefficients), and for the important class of étale groupoids in [4].

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